# MATH 426/529: Extremal Combinatorics

## Term Test 1

October 15, 2026

Name: Solutions

Student number: _____

**Instructions.**

- This exam is 80 minutes.
- There are 36 marks available.
- No collaboration is allowed.
- Write complete solutions. Unless otherwise stated, justify your answers.

| Question | Marks available | Marks earned |
|---|---|---|
| 1 | 6 | |
| 2 | 8 | |
| 3 | 11 | |
| 4 | 11 | |
| Total | 36 | |

## Question 1. [6 marks]

(a) Provide an example of a sunflower $\mathcal{F} \subseteq 2^{\mathbb{N}}$ with four petals.

(b) What does it mean for a set system $\mathcal{F} \subseteq 2^{[n]}$ to *shatter* a set $S \subseteq [n]$?

(c) What does it mean for a set $A = \{a_1, \ldots, a_k\}$ to come before another set $B = \{b_1, \ldots, b_k\}$ in *colexicographic order* on $\binom{\mathbb{N}}{k}$?

(a) For example, $\mathcal{F} = \{\{1\}, \{2\}, \{3\}, \{4\}\}$ is a sunflower with 4 petals and core $\emptyset$.

(b) It means that, for all $T \subseteq S$, there exists $A \in \mathcal{F}$ such that $S \cap A = T$.

(c) Assuming that $a_1 < \cdots < a_k$ and $b_1 < \cdots < b_k$, it means that $a_i < b_i$ where $i = \max\{j : a_j \neq b_j\}$.

## Question 2. [8 marks]

State each of the following theorems from lectures.

(a) The Erdős–Ko–Rado Theorem.

(b) The Sauer–Shelah Lemma.

(c) The LYM Inequality.

(d) The Harris–Kleitman Inequality.

(a) Let $n$ and $k$ be positive integers such that $k \leq n/2$. If $\mathcal{F} \subseteq \binom{[n]}{k}$ is intersecting, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

(b) Let $n \geq 1$ and $0 \leq k \leq n$ be integers. If $|\mathcal{F}| > \sum_{i=0}^{k-1} \binom{n}{i}$, then the VC-dimension of $\mathcal{F}$ is at least $k$.

(Or, I would also accept: Every finite set system $\mathcal{F}$ shatters at least $|\mathcal{F}|$ sets).

(c) If $\mathcal{A} \subseteq 2^{[n]}$ is an antichain, then

$$\sum_{A \in \mathcal{A}} \frac{1}{\binom{n}{|A|}} \leq 1.$$

(d) If $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$ are downsets, then

$$\frac{|\mathcal{A} \cap \mathcal{B}|}{2^n} \geq \frac{|\mathcal{A}|}{2^n} \cdot \frac{|\mathcal{B}|}{2^n}.$$

## Question 3. [11 marks]

(a) Let $\mathcal{F}$ be the collection of all subsets of $\mathbb{R}$ that can be written as a union of at most 2 intervals. Prove that the VC-dimension of $\mathcal{F}$ is equal to 4.

(b) Let $\mathcal{F} \subseteq 2^{[n]}$ be a downset. Prove that the VC-dimension of $\mathcal{F}$ is equal to

$$\max\{|A| : A \in \mathcal{F}\}.$$

(c) Given an examples of two functions $f, g : \mathbb{N} \to \mathbb{N}$ such that $f = \Theta(g)$ but there does not exist a constant $c > 0$ such that $f \sim c \cdot g$.

*Hint:* Recall that two functions $\phi, \psi : \mathbb{N} \to \mathbb{N}$ satisfy $\phi \sim \psi$ if $\lim_{n\to\infty} \frac{\phi(n)}{\psi(n)} = 1$.

(a) First, let us show that it is at least four.
Let $S = \{1,2,3,4\}$. We show that $\mathcal{F}$ shatters $S$.
It is easy to find a set in $\mathcal{F}$ whose intersection with $S$ is $\emptyset$. It is also easy to isolate each singleton; eg. $S \cap [1.5, 2.5] = \{2\}$. Any pair can also be isolated using a union of two intervals. Any subset of $S$ of size 3 contains two consecutive points, which can be covered by one interval, and the other point can be covered by a second interval. Eg $S \cap ([0, 2.5] \cup [3.5, 4.5]) = \{1,2,4\}$.
Also, $S \cap [-10, 10] = S$. Now, to see that it is exactly four,
Consider any set $S = \{x_1, x_2, x_3, x_4, x_5\} \subseteq \mathbb{R}$ with $x_1 < x_2 < x_3 < x_4 < x_5$.
Then there can be no set $A \in \mathcal{F}$ such that $S \cap A = \{x_1, x_3, x_5\}$ because any interval containing $x_1$ and $x_3$ must contain $x_2$, any interval containing $x_3$ and $x_5$ must contain $x_4$ and any interval containing $x_1$ and $x_5$ contains all of $S$. So, the VC-dim is 4.

(b) Let $A \in \mathcal{F}$ such that $|A| = \max\{|A| : A \in \mathcal{F}\}$.
Since $\mathcal{F}$ is a downset, every subset of $A$ is in $\mathcal{F}$.
Also, clearly, if $B \subseteq A$, then $A \cap B = B$. So $\mathcal{F}$ shatters $A$.

Now, suppose $S \subseteq [n]$ such that $|S| > \max\{|A| : A \in \mathcal{F}\}$.

Then, for any $A \in \mathcal{F}$, we have $|A \cap S| \leq |A| < |S|$.

So there is no set $A \in \mathcal{F}$ such that ~~$A \in \mathcal{A}$~~
$A \cap S = S$ and so $\mathcal{F}$ does not shatter $S$.

(c) Let $f(n) = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 2 & \text{if } n \text{ is even} \end{cases}$

$$g(n) = 1.$$

Then $\frac{1}{2} f(n) \leq g(n) \leq f(n)$ for all $n$.
therefore $g = \Theta(f)$. However, the
limit $\lim_{n \to \infty} \frac{f(n)}{g(n)}$ does not exist.

## Question 4. [11 marks]

(a) Let $a_1, \ldots, a_n \geq 1$, and let $b \in \mathbb{R}$. Define

$$\mathcal{F} = \left\{ S \subseteq [n] : b \leq \sum_{i \in S} a_i < b + 1 \right\}.$$

Prove that $|\mathcal{F}| \leq \binom{n}{\lfloor n/2 \rfloor}$.

(b) Let $\mathcal{F} \subseteq \binom{[n]}{3}$ such that $|\mathcal{F}| = 11$. Prove that

$$|\partial \mathcal{F}| \geq 12.$$

(a) We show that $\mathcal{F}$ is an antichain, from which the result will follow by Sperner's Thm.

If $\mathcal{F}$ is not an antichain, then $\exists A, B \in \mathcal{F}$ such that $A \subsetneq B$. So,

$$\sum_{i \in B} a_i = \sum_{i \in A} a_i + \sum_{i \in B \setminus A} a_i$$

$$\geq \sum_{i \in A} a_i + 1 \quad \left(\text{because } a_i \geq 1 \text{ for all } i \text{ and } B \setminus A \neq \emptyset\right)$$

Since $A \in \mathcal{F}$, we know $\sum_{i \in A} a_i \geq b$.

But then $\sum_{i \in B} a_i \geq b+1$ which contradicts the assumption that $B \in \mathcal{F}$.

(b) By Kruskal-Katona, $|\partial \mathcal{F}|$ is minimized when $\mathcal{F}$ is the first 11 sets in colex.

The 11th set is $\{a_1, a_2, a_3\}$ where $a_1 < a_2 < a_3$ and $11 = 1 + \binom{a_1-1}{1} + \binom{a_2-1}{2} + \binom{a_3-1}{3}$.

Since $\binom{5}{3} = 10 < 11$, we must have $a_3 \geq 6$.

We have $11 = 1 + \binom{1-1}{1} + \binom{2-1}{2} + \binom{6-1}{3}$ and

So $a_1 = 1$, $a_2 = 2$, $a_3 = 6$.

$$\text{So, } |\partial \mathcal{F}| \geq \Big| \underbrace{\binom{[5]}{3}}_{\text{first 11 sets in colex}} \cup \{\{1,2,6\}\} \Big|$$

$$= \left| \binom{[5]}{2} \cup \{\{1,6\}, \{2,6\}\} \right|$$

$$= 12.$$